Dynamics of a relativistic Rankine vortex for a two-constituent superfluid in a weak perturbation of cylindrical symmetry

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Abstract

From a recent study of a stationary cylindrical solution for a relativistic two-constituent superfluid at low temperature limit, we propose to specify this solution under the form of a relativistic generalisation of a Rankine vortex (Potential vortex whose the core has a solid body rotation). Then we establish the dynamics of the central line of this vortex by supposing that the deviation from the cylindrical configuration is weak in the neighbourhood of the core of the vortex. In "stiff" material the Nambu-Goto equations are obtained.

1 Introduction

First works in phenomenological relativistic superfluidity were undertaken by Rothen [1], Dixon [2] and Israel [3, 4]. The framework of this study is the two-constituent relativistic superfluid dynamics derived by the convective variational approach. This description, a specialisation of a general formalism developed by Carter [5, 6] , can be considered as a relativistic extention of the standard non dissipative Landau superfluid dynamics [7] and give the same result [8] as that of Khalatnikov and Lebedev [9].

The purpose of this work is to find the relativistic dynamics of the central line of a vortex in a "cool" superfluid when it deviates weakly of the stationarity and of the cylindrical symmetry. Carter and Langlois [10, 11] have introduced the denomination cold superfluid at zero temperature and cool superfluid at low temperature when the only excitations are the phonons (no rotons).

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In a cool superfluid, the precedent authors [11] find the generic vortex solution (static cylindrically symmetric solution) in Minkowski space. In this solution the irrotational covector μ_ν which describes the superfluid constituent becomes spacelike near the center of the vortex. If we attribute some physical reality to this covector, we must cure this pathological situation. One means is to prevent the superfluid constituent to have access to the neighbourhood of the center of the vortex. This can be obtained by imposing a constraint on the integrals of motion of the vortex solution.The second constituent represented by the entropy current vector s^σ which contains all the excitations of the superfluid,posseses the motion of a solid body. So the current s^σ becomes spacelike far of the center of the vortex. This problem has been solved [11] by a natural cut off radius of the vortex. This situation incites us to specify the static cylindrical solution under the form of relativistic generalisation of a Rankine vortex [12] (potential vortex whose the core has a solid body rotation)

As we shall see in the following we can suppose that the core of this vortex is thin and contains all the excitations. This situation is not without reminding us superconductor of type II where the normal conductor is confined in the tubes of magnetic flux. However let us note that it is not a real separation of the excitations from the superfluid part since this feature arrives in the particular structure of a vortex.

Once defined the generalisation of the Rankine vortex, we shall suppose that the central line of this vortex has a small deviation from the straight configuration. So we adopt the assumption that locally on the boundary of the thin core of the vortex, μ_ν at the exterior and s^σ in the interior are those describing the straight vortex.Then, expanding the equations of motion in function small parameters caracteristic of the curvature and the thickness of the core, the lower order which is the equation of the straight vortex, is automatically verified.The following order gives approximatly the dynamical equation of the central line. This method has been transposed from a study of the dynamics of a self-gravitating thin cosmic string [13].

The plan of the paper is the following. In section 2 and 3, we shall review the essential ingredients of the cool superfluid [10] and of its stationnary and cylindrical solution [11] which are the base of this work. In section 4 we define the generalisation of a Rankine vortex. In section 5 we obtain the dynamical equation of the central line. In stiff matter it is reduced to the Nambu-Goto dynamics. Finally in section 6 we give a conclusion.

2 Cool superfluid

The equation of the relativistic superfluid mecanics in the non dissipative limit can be derived [8, 10] from a lagrangian $\mathcal{L}(s^\sigma, \mu_\sigma)$, where s^σ is the entropy current and μ_ν is the superfluid momentum covector which is the gradient

$$\mu_\sigma = \hbar \nabla_\sigma \varphi \tag{1}$$

of the superfluid phase scalar φ. The variation of $\mathcal{L}$,

$$d\mathcal{L} = \Theta_\sigma ds^\sigma - n^\sigma d\mu_\sigma, \tag{2}$$

yields the thermal momentum covector Θ_σ associated with the the entropy current s^σ and the particle current n^σ associated with the superfluid momentum covector μ_σ.

The equations of motion are (1) and

$$s^\rho(\nabla_\rho\Theta_\sigma - \nabla_\sigma\Theta_\rho) = 0, \tag{3}$$

$$\nabla_\rho n^\rho = 0, \tag{4}$$

$$\nabla_\rho s^\rho = 0. \tag{5}$$

The energy-momentum tensor corresponding to the lagrangian $\mathcal{L}$ has the form

$$T^{\rho\sigma} = n^\rho\mu^\sigma + s^\rho\Theta^\sigma + \Psi g^{\rho\sigma}, \tag{6}$$

where

$$\Psi = \mathcal{L} - \Theta_\rho s^\rho \tag{7}$$

is the pressure function. Equations (1),(3),(4) and (5) entail the conservation of the energy-momentum tensor $T^{\rho\sigma}$.

The lagrangian $\mathcal{L}$ is built with the three available scalar quantities

$$c^2 s^2 = -s_\rho s^\rho, \tag{8}$$

$$c^2 y^2 = -\mu_\rho s^\rho, \tag{9}$$

$$c^2\mu^2 = -\mu_\rho\mu^\rho. \tag{10}$$

The variables n^ρ and Θ_ρ defined by (2) can be written in terms of μ_ρ and s^ρ by

$$n^\rho = \Phi^2(\mu^\rho - As^\rho) \quad , \quad \Theta_\rho = \Phi^2(Ks_\rho + A\mu_\rho) \tag{11}$$

where Φ^2, K and A are given by

$$c^2\Phi^2 = 2\frac{\partial\mathcal{L}}{\partial\mu^2} \quad , \quad c^2\Phi^2 K = -2\frac{\partial\mathcal{L}}{\partial s^2} \quad , \quad c^2\Phi^2 A = -\frac{\partial\mathcal{L}}{\partial y^2}. \tag{12}$$

The masse current is defined [14] as

$$j^\rho = j^\rho_N + j^\rho_S \tag{13}$$

where

$$j^\rho_N = -m\Phi^2 As^\rho \tag{14}$$

is interpreted, according to the usual terminology, as the normal mass current and

$$j^\rho_S = m\Phi^2\mu^\rho \tag{15}$$

as the superfluid current.

In the cool regime in which the entropy is associated just to phonons (not rotons) the lagrangian is given [10] by

$$\mathcal{L} = P(\mu^2) - 3\psi(s^2, \mu^2, y^2). \tag{16}$$

P represents the pressure in the cold limit in which the entropy vanishes and ψ represents the general pressure of the phonon gaz which is given by

$$\psi = \tilde{\hbar}\frac{1}{3}c_s^{-\frac{1}{3}}(c^2 s^2 + (c_s^2 - c^2)\frac{y^4}{\mu^2})^{\frac{2}{3}} \tag{17}$$

with $\tilde{\hbar} \simeq 0.99\hbar$ and where c_s is the cold sound speed determined by P, according to

$$\frac{c^2}{c_s^2} = \mu\frac{d\mu}{dP}\frac{d^2P}{d\mu^2}. \tag{18}$$

3 Sationary and cylindrically symmetric solutions

In a Minkowski space equiped with cylindrical coordinates

$$ds^2 = -c^2dt^2 + dz^2 + dr^2 + r^2d\phi^2, \tag{19}$$

this class of solutions [11] is characterized by helical current vectors

$$n^\rho = \nu(r)(1, v(r), 0, \omega(r)), \tag{20}$$

$$s^\rho = \sigma(r)(1, V, 0, \Omega). \tag{21}$$

These currents have no radial components.The quantities $v(r)$ and $\omega(r)$ are respectively the translation velocity and the angular velocity of the particle current. They are radially dependent. Whereas the homologous quantities V and Ω of the entropy constituent are constant.This last requirement express that the “normal” constituent has a rigid motion. This condition prevent dissipation by inevitable small vicosity which, for non-rigid motion, would appear in the “normal” flow of a realistic model.

The class of such helical flow includes the more usual subclass of circular flow for which the longitudinal translation velocities v and V are zero.

The resolution of the equations gives in addition to Ω and V, four other constants of motion:

$$\overline{\Theta} = \overline{k}^\rho\Theta_\rho \tag{22}$$

with

$$\overline{k}^\rho = \sigma^{-1}s^\rho. \tag{23}$$

$\overline{\Theta}$ is interpreted as an effective temperature measured in the corotating frame of the “normal” flow.

The components of the superfluid momentum covector

$$\mu_\sigma = (-E, L, 0, M), \tag{24}$$

where the three constants E, L, M are respectively interpreted as the energy, the longitudinal momentum and the angular momentum per particle. So there are six integral constants $\overline{\Theta}$, V, Ω,E, L and M which is sufficient to solve the problem which depends in reality of six independent field components. Let us note that by Lorentz transformation along z we can choose $L = 0$ or $V = 0$.

The scalar quantities μ, s, y are calculated from (21) and (24):

$$c^2\mu^2 = \frac{E^2}{c^2} - L^2 - \frac{M^2}{r^2}, \tag{25}$$

$$s^2 = \sigma^2(r)(1 - \frac{V^2}{c^2} - \frac{\Omega^2 r^2}{c^2}), \tag{26}$$

$$c^2 y^2 = \sigma(r)(E - VL - \Omega M), \tag{27}$$

and in the cool regime the results recorded in section 2 allow to obtain an explicit solution [11].

4 Relativistic vortex of Rankine

If a physical meaning is attributed to various currents and momenta n^μ , s^μ, μ_ν, Θ_ν, introduced in this theory, they must never become spacelike. However in the stationary cylindrical symmetric solution the expressions (25) and (26) show that μ_ν and s^ν can become spacelike for different domains of the radial coordinate r.

For s^μ the validity of the solution must be limited to a finite range of r. This question has been treated by Carter and Langlois [11] in order to can describe arrays of vortices in neutron stars.

For μ_ν the question is also important since the momentum becomes spacelike in the neighbourood of $r = 0$ which is the center of the vortex.

In the non relativistic theory, the rotation of the superfluid can be satisfactorily explained by assuming that it is threaded by a series of parallel straight vortex lines since there is no principle of limited velocity in Newtonian theory. (There is of course a physical limit which depends of the effective "thickness of the line" but this is not a question of principle.)

On contrary, in the preceeding relativistic theory, the stationary cylindrical solution cannot competly describe a vortex without some restriction, since, for any given solution, the relation (25) determines a radius $r = a_0$ by

$$\frac{E^2}{c^2} - L^2 - \frac{M^2}{a_0^2} = 0 \tag{28}$$

inside which μ_ρ becomes spacelike.Therefore the relativity imposes a condition which does'nt exist in non relativistic theory of superfluid.

It is interesting to have an estimation of the minimum radius a_0 in HeII. We suppose that we have a not too fast circular motion ($L = 0$) and that the vortex is quantified ($M = n\hbar$). From (28), for one quantum we have

$$a_0 = \frac{Mc}{E} \simeq \frac{\hbar c}{4mc^2} \simeq 5.10^{-15}\,\mathrm{cm} \tag{29}$$

There is experimental evidence [15] that the typical core radius r_0 of a vortex in HeII is about 10^{-8}cm. So the limit a_0 is probably never reached.

In the frame of this purely classical relativistic fluid theory, the most natural way to overcome the theoretical difficulty of the spacelike current is to suppose that the normal part of the fluid (s^ρ, Θ_ρ) is concentrated in the neighbourhood of the center and constitutes a solid core whereas the the superfluid part (n^ρ, μ_ρ) is rejected around the core. This constitutes a relativistic generalisation of a Rankine vortex in which the core has a rigid body rotation and the outside has a circular potential flow.

Such a solution looks as it separates the normal component inside the core from the superfluid component outside the core. This seems a contradiction with the usual interpretation where it is stated that the two fluids cannot be physically separeted. But in our case the fluids are not physically separated since they are binded in the same vortex whose the configuration is imposed by the relativity. This configuration can be compared to the situation where two reservoirs are connected by a superleak. A superleak is a channel through which the superfluid can flow, but not the normal fluid.This device can be interpreted as a separation between the two components since there is only superfluid in the channel. As we shall see below, the comparison is still yet more striking. Let us examine the question of the temperature. B.Carter and D.Langlois attributed an effective temperature $\overline{\Theta} = \overline{k}^\rho \Theta_\rho$ that is uniform throughout all the fluid. But in the above configuration, the temperature which is associated to the normal fluid (phonon) would have to concerne only the core whereas outside the core the temperature would be null. Can such a solution exist in a stationary system? To answere this question let us look again at the HeII experiments. We know that a temperature gradient can be set up between two volumes of bulk HeII provided that they are connected by a superleak [15]. There is also a gradient of pression to maintain the equilibrium situation, and we have

$$\frac{\Delta \mathrm{Pression}}{\Delta \mathrm{Temperature}} = \text{entropy per unit of volume} \tag{30}$$

In a relativistic vortex the relativity acts at the frontier of the core as a superleak, it prevents the superfluid to get in the center of the vortex, similarily the superleak clamps the normal fluid, and as it will be showed below, at the frontier of the core there is a relation like (30).

Let us precise the equations of this relativistic Rankine model.The difficulty is to exclude the superfluid part of the center of the vortex. In general the lagrangian (16), (17) does

not allows this, since ψ merges μ_ρ and s^ρ. But there is a relation among the constants of motion in which μ_ρ desapears of ψ :

$$\overline{E} = E - VL - \Omega M = 0 \tag{31}$$

In this case $y^2 = 0$ and ψ is reduced to

$$\psi = \frac{\tilde{\hbar}}{3} c_s^{\frac{-1}{3}} (c^2 s^2)^{\frac{2}{3}} \tag{32}$$

If we suppose that c_s is a constant, ψ becomes a fonction of s only.

The lagrangian becomes:

$$\mathcal{L} = P(\mu^2) - 3\psi(s^2) \tag{33}$$

where $P(\mu^2)$ is the pression of the cold limit (without excitation).

$P(\mu^2)$ rules the exterior of the vortex which is a cold potential fluid. Since c_s is supposed constant the relation (18) gives an expression for P

$$P = a\mu^{\frac{c^2}{c_s^2}+1} + b, \tag{34}$$

where a and b are constants of integration.

The relation (11) between currents and moments becomes

$$n^\rho = \Phi^2(\mu^2)\mu^\rho \tag{35}$$

where Φ^2 is given from (12) and (33) by

$$c^2\Phi^2 = 2\frac{\partial P(\mu^2)}{\partial \mu^2} \tag{36}$$

and μ^ρ by

$$\mu^\rho = (\frac{E}{c^2}, L, 0, \frac{M}{r^2}). \tag{37}$$

The equation of motion

$$\nabla_\sigma(\Phi^2(\mu^2)\mu^\sigma) = 0 \tag{38}$$

is verified.

The core of the vortex is ruled by the part of the lagrangian $-3\psi(s^2)$ where ψ is the pression of the excitations (phonon gaz). Let us note that the relation (31) imposes a radius $r_0' \leq a_0$ inside which s^ρ is not spacelike. The equality is obtained in the circular motion. The relation between currents and moments is reduced to

$$\Theta_\rho = \Phi^2 K s_\rho \tag{39}$$

with

$$c^2\Phi^2 K = 2 \times 3\frac{\partial \psi}{\partial s^2} = \frac{4}{3}\tilde{\hbar} {c_s}^{-\frac{1}{3}} c^{\frac{4}{3}} \left(s^2\right)^{-\frac{1}{3}} \tag{40}$$

and

$$s_\rho = \sigma(r)(-c^2, V, 0, \Omega r^2) \tag{41}$$

The temperature of the core is given from (22) and (23) by

$$\overline{\Theta} = \frac{4}{3}\tilde{\hbar} c_s^{-\frac{1}{3}} \sigma^{\frac{1}{3}} c^{\frac{4}{3}} \left(1 - \frac{V^2}{c^2} - \frac{\Omega^2 r^2}{c^2}\right)^{\frac{2}{3}} \tag{42}$$

hence $\sigma(r)$ is obtained in this case by

$$\sigma(r) = \frac{c_s}{\left(\frac{4}{3}\tilde{\hbar}\right)^3 c^4} \frac{\overline{\Theta}^3}{\left(1 - \frac{V^2}{c^2} - \frac{\Omega^2 r^2}{c^2}\right)^2} \tag{43}$$

The equations (3) and (5) are the equations of motion in the core which has a rigid motion (velocity V along z and rotation Ω around z).

In this stationary vortex solution, there is a paradoxical gradient of temperature at the boundary of the core. The temperature is $\overline{\Theta}$ in the core and zero at the exterior. This is not normally an equilibrium situation. But let us examine if it is not possible to have a relation like (30) which characterises an equilibrium in a superfluid in presence of a superleak.

Let us remarck that

$$\frac{\psi}{\overline{\overline{\Theta}}} = \frac{1}{4}\sigma(r) \tag{44}$$

where $\sigma(r)$ is the proper density of entropy in the core. If we choose on the frontier r_0 of the core

$$P(\mu^2) = -\psi(s^2) \tag{45}$$

we obtain on r_0

$$\frac{\Delta\text{Pression}}{\Delta\text{Temperature}} = \frac{\psi - P}{\overline{\Theta} - 0} = \frac{\sigma}{2} \tag{46}$$

where $\frac{\sigma}{2}$ is the mean value of the proper density of entropy at the frontier of the core. Let us note that (45) can be obtained from (34) by choosing the constant a positive and b negative. $P(\mu^2)$ is a tension in the neighbourhood of the core.

We think that analogy with a superleak is strong enough to accept the difference of pression and temperature at the boundary between the two fluids.

5 Dynamics of long waves along the core of the vortex ($c = 1$)

We consider now a little deformation of the relativistic Rankine vortex described above. Its central line has an arbitrary shape but so that its curvature is weak. The section of its core stays circular with a little radius r_0. Its motion is slow. We can imagine waves of small amplitude along a straight vortex.

One introduces the local coordinates system (τ^A, ρ^a) attached to the central line of the vortex:

$$x^\mu = X^\mu(\tau^A) + \rho^a N_a^\mu(\tau^A) \tag{47}$$

The Minkowskian coordinates x^μ are expressed in function of the two coordinates $\tau^A = (\tau^0, \tau^3)$ of the world sheet sweeped by the central line, and of two coordinates $\rho^a = (\rho^1, \rho^2)$ pointing in a direction orthogonal to the world sheet along the two orthonormal vectors N_a^μ. We introduce also polar coordinates:

$$\rho^1 = r\cos\phi \quad , \quad \rho^2 = r\sin\phi. \tag{48}$$

In this problem we need to define a typical length, choosed unity by convenience, in order to characterize the spacetime neighbourhood of a point of the central line in which we shall study the vortex. This typical length is an estimate of the radius of the vortex. We shall suppose that the characteristic length l on wich we have a change in the tangent plane of the world sheet is large and that the radius of the core r_0 is small

$$l \gg 1 \quad , \quad r_0 \ll 1$$

Let us note that we have two typical lenght l and r_0.

The vetors N_a^μ perpendicular to the world sheet and the tangent vectors $\frac{\partial X^\mu}{\partial \tau^A}$ have little change on a length unity. We can express this fact by

$$N_a^\mu = N_a^\mu(\frac{\tau^A}{l}) \quad , \quad \frac{\partial X^\mu}{\partial \tau^A} = X^\mu_{,A}(\frac{\tau^A}{l}) \tag{49}$$

hence

$$\partial_A N_a^\mu = \frac{1}{l} N^\mu_{a,A} \quad , \quad \partial_B X^\mu_{,A} = \frac{1}{l} X_{,AB}. \tag{50}$$

The metric $g_{\alpha\beta}$ in the system of coordinates(τ^A, ρ^a) can be expressed by

$$g_{AB} = \gamma_{AB} + 2K_{aAB}\rho^a + \left(K^D_{bA}K_{aBD} + \delta_{ab}\omega_A\omega_B\right)\rho^a\rho^b \tag{51}$$

$$g_{Ab} = \rho^a \epsilon_{ab}\omega_A \tag{52}$$

$$g_{ab} = \delta_{ab} \tag{53}$$

wherein we recognise the induced metric of the world sheet

$$\gamma_{AB} = X^\mu_{,A}X^\nu_{,B}\eta_{\mu\nu} = O(\frac{1}{l^0}), \tag{54}$$

the extrinsic curvature

$$K_{aAB} = \frac{1}{l}\eta_{\mu\nu}N^\mu_{a,A}X^\nu_{,B} = O(\frac{1}{l}) \tag{55}$$

and the twist defined by

$$\epsilon_{ab}\omega_A = \frac{1}{l}\eta_{\mu\nu}N^\mu_{a,A}N^\nu_b = O(\frac{1}{l}). \tag{56}$$

Some other quantities will be useful:

$$\gamma = \det \gamma_{AB}, \tag{57}$$

the mean curvature

$$K_a = K_{aAB}\gamma^{AB}, \tag{58}$$

$$g = \det g_{\alpha\beta} = \gamma D^2 \tag{59}$$

where

$$D = 1 + K_a\rho^a + \frac{1}{2}(K_aK_b - K^A_{aB}K^B_{bA})\rho^a\rho^b, \tag{60}$$

and

$$g^{AB} = \gamma^{AB} - 2K^{AB}_a\rho^a + O(\frac{r_0^2}{l^2}) \tag{61}$$

In the following we suppose that there is no twist in the deformation so that the cross terms of the metric cancel.

It is natural to adopt the assumption that in the neighbourhood of the core whose the linear dimentions are some unities of r_0, the solution coincide at the lower order with the solution of a stationary cylindrical vortex described in the preceding section, that is with (21) for the interior and (24) for the exterior. In Minkowskian coordinates, they are rewritten

$$\mu_\nu = (-E, L, -\frac{y}{r^2}M, \frac{x}{r^2}M), \tag{62}$$

$$s^\rho = \sigma(r)(1, V, -y\Omega, x\Omega), \tag{63}$$

where $\sigma(r)$ is defined in function of the constant $\overline{\Theta}$ by (43).

We can choose the coordinates τ^A on the world sheet so that γ_{AB} takes the conformally flat form:

$$\gamma_{AB} = F(\frac{\tau^A}{l})\eta_{AB} \tag{64}$$

In a point of the world sheet chosen as origine ($\tau^A = 0$) we can fix $\gamma_{AB} = \eta_{AB}$ so that in the neighbourhood we have

$$F(\frac{\tau^A}{l}) = 1 + \frac{\tau^D}{l}f_D(\frac{\tau^A}{l}). \tag{65}$$

The holonomic base ∂_A, ∂_a is orthogonal but not orthonormal so that the identification of the solution with the stationary cylindrical solution must take into account the norm $\sqrt{F}$ of ∂_A, hence

$$\mu_\sigma = (-\sqrt{F}E, \sqrt{F}L, -\frac{\rho^2}{r^2}M, \frac{\rho^1}{r^2}M), \tag{66}$$

$$s^\sigma = \sigma(r)(\frac{1}{\sqrt{F}}, \frac{V}{\sqrt{F}}, -\rho^2\Omega, \rho^1\Omega), \tag{67}$$

with

$$\sigma(r) = \frac{\overline{\Theta}^3 c_s}{\left(\frac{4}{3}\tilde{\hbar}\right)^3}\left(1 - V^2 - r^2\Omega^2\right)^{-2} \tag{68}$$

are the solutions expressed in the holonomic base(∂_A, ∂_a). When expressed in the orthonormal tetrad $(\frac{\partial_A}{\sqrt{F}}, \partial_a)$ we can see that these solutions coincide with the solutions (62) (63) of the cylindrical vortex.

We shall write the equation of motion (38) for the exterior and (3), (5) for the interior on the frontier of the core and expand these equations in power of the small parameter l^{-1};. The zero order which is the solution of the straight vortex vanish identically. In this expansion appears also the second small parameter r_0 .

For this purpose it will be useful to rewrite the expression (66) and (67) in a more condensate form:

$$\mu_\sigma = (\overline{\mu}_A \sqrt{F}\,,\, \mu_a), \tag{69}$$

$$s^\rho = \sigma(r)(\frac{\overline{s}^A}{\sqrt{F}}\,,\, \overline{s}^a), \tag{70}$$

where

$$\overline{\mu}_A = (-E, L) \quad , \quad \overline{s}^A = (1, V) \tag{71}$$

and

$$\mu_a = (-\frac{\rho^2}{r^2}M\,,\, \frac{\rho^1}{r^2}M) \quad , \quad \overline{s}^a = (-\rho^2\Omega\,,\, \rho^1\Omega). \tag{72}$$

μ_a and $\overline{s}^a$ are respetively of order r_0^{-1} and r_0.

We must calculate some quantities.Using (61), (64), (65), the contravariant components μ^α are given by

$$\mu^A = g^{AB}\mu_B = \eta^{AB}\overline{\mu}_B - \frac{1}{2}\eta^{AB}\frac{\tau^D}{l}f_D\overline{\mu}_B - 2K_a^{AB}\rho^a\overline{\mu}_B + \ldots \tag{73}$$

$$\mu^a = \delta^{ab}\mu_b. \tag{74}$$

In (73) the second term is of order $\frac{\tau^D}{l} \leq \frac{1}{l}$, the third term is of order $\frac{r_0}{l}$. The ellipse designates smaller terms; we shall follow this convention below. We can express

$$\mu^2 = -g^{\alpha\beta}\mu_\alpha\mu_\beta = -g^{AB}\mu_A\mu_B - \frac{M^2}{r^2} \tag{75}$$

in function of the corresponding quantity (25) of the straight vortex which will be renamed

$$\mu_0^2 = -\eta^{AB}\overline{\mu}_A\overline{\mu}_B - \frac{M^2}{r^2}. \tag{76}$$

We obtain

$$\mu^2 = \mu_0^2 + 2K_a^{AB}\rho^a\overline{\mu}_A\overline{\mu}_B + \ldots \tag{77}$$

and

$$\Phi^2(\mu^2) = \Phi^2(\mu_0^2) + 2K_a^{AB}\rho^a\overline{\mu}_A\overline{\mu}_B\frac{\partial\Phi^2}{\partial\mu^2}(\mu_0^2) + \ldots. \tag{78}$$

We have also

$$s^2 = -g_{\alpha\beta}s^\alpha s^\beta = \sigma^2(r)(1 - V^2 - \Omega^2 r^2 - \delta^2 + \ldots), \tag{79}$$

with

$$\delta^2 = 2F^{-1}K_{aAB}\rho^a\overline{s}^A\overline{s}^B = O(\frac{r_0}{l}). \tag{80}$$

We can express (79) in function of the corresponding quantity (26) of the straight vortex renamed s_0^2:

$$s^2 = s_0^2 + \sigma^2(r)\delta^2 + \ldots. \tag{81}$$

It is possible now to expand the momentum (39), (40), namely $\Theta_\rho = d(s^2)^{-\frac{1}{3}}s_\rho$ with $d = \frac{4}{3}\tilde{\hbar}c_s^{-\frac{1}{3}}$ in the form

$$\Theta_\rho = d\left((s_0^2)^{-\frac{1}{3}} - \frac{1}{3}\sigma^2(r)\delta^2(s_0^2)^{-\frac{4}{3}} + \ldots\right)\sigma(r)\left(g_{AB}\frac{\overline{s}^B}{\sqrt{F}}\,,\,\overline{s}_a\right) \tag{82}$$

With

$$g_{AB}\frac{\overline{s}^B}{\sqrt{F}} = \sqrt{F}\eta_{AB}\overline{s}^B + \frac{2}{\sqrt{F}}\rho^a K_{aAB}\overline{s}^B + \ldots \tag{83}$$

We can now expand the equations of motion on the frontier $r = r_0$ of the core of the vortex. The exterior equation (38) in coordinates (τ^A, ρ^a)

$$\frac{1}{FD}\partial_\sigma\left(FD\Phi(\mu^2)\mu^\sigma\right) = 0 \tag{84}$$

gives

$$\frac{1}{FD}\left(\Phi^2(\mu_0^2)\eta^{AB}\frac{1}{2l}f_A\overline{\mu}_B + \Phi^2(\mu_0^2)K_a\mu_b\delta^{ab} + \frac{\partial\Phi^2}{\partial\mu^2}(\mu_0^2)\overline{K}_a\mu_b\delta^{ab}\right) + \ldots = 0 \tag{85}$$

with

$$\overline{K}_a = 2K_a^{CD}\overline{\mu}_C\overline{\mu}_D \tag{86}$$

In this derivation we have used that for any function $G(r)$:

$$\partial_a\left(G(r)\delta^{ab}\mu_b\right) = 0$$

In the equation (85) the first term is of order $\frac{1}{l}$, the second and the third are in $\frac{1}{lr_0}$. So we shall retain only the second and third terms in $\frac{1}{lr_0}$ which are the leading terms of the expantion (85).

Before to continue this calculation let us have a look on the interior equations (3) and (5) which in coordinates (τ^A, ρ^a) are written

$$s^\rho(\partial_\rho\Theta_\sigma - \partial_\sigma\Theta_\rho) = 0 \tag{87}$$

$$\frac{1}{FD}\partial_\sigma(FDs^\sigma) = 0 \tag{88}$$

Introducing (82) and (83)in these equations we can express the expansion of these equations. Calculation which is not useful to reproduce, shows that after the terms of zero

order in $\frac{1}{l}$ which cancel identically from the result of the straight vortex, the leading terms are $\frac{1}{l}$ and $\frac{r_0}{l}$. These terms are negligeable before the second and third terms (in $\frac{1}{lr_0}$) of equation (85). Hence the interior equations do not contribute at the same order and can be forgotten in the resolution of this problem. Therefore we are left with the leading term of the equation (85):

$$\left(\Phi^2(\mu_0^2)K_a + \frac{\partial\Phi^2}{\partial\mu^2}(\mu_0^2)\overline{K}_a\right)\mu^a = 0 \tag{89}$$

This equation is written on the frontier $r = r_0$ of the core; μ^a depends of the polar angle ϕ. Since the equation (89) must be verified for all the polar angles when we turn around the core, we obtain finally the equation:

$$\Phi^2(\mu_0^2)K_a + \frac{\partial\Phi^2}{\partial\mu^2}(\mu_0^2)\overline{K}_a = 0 \tag{90}$$

Since the radius r_0 of the core is supposed small, this equation gives a good approximation of the equation of motion of the central line.

The equation (16) can be written

$$\frac{d\Phi^2}{d\mu^2} = \left(\frac{1}{c_s^2} - 1\right)\frac{\Phi^2}{2\mu^2}, \tag{91}$$

hence the equation (90) is written:

$$K_a + \frac{1}{2\mu_0^2}\left(\frac{1}{c_s^2} - 1\right)\overline{K}_a = 0. \tag{92}$$

In the stiff material where $c_s = 1$, the central line has the Nambu-Goto dynamics:

$$K_a = 0 \tag{93}$$

Let us note an interesting form of the equations of motion (90):

$$K_a^{AB}S_{AB} = 0 \tag{94}$$

where

$$S_{AB} = \frac{\partial\Phi^2}{\partial\mu^2}(\mu_0^2)\overline{\mu}_A\overline{\mu}_B + \frac{1}{2}\Phi^2(\mu_0^2)\eta_{AB} \tag{95}$$

We shall study the linear approximation of the equation (94). Let us project the extrinsic curvature K_{aAB} in the Minkowski space:

$$K_{AB}^{\mu} = -\delta^{ab}K_{aAB}N_b^{\mu} = \partial_A\partial_B X^{\sigma}\left(\delta_{\sigma}^{\mu} - \eta_{\rho\sigma}\eta^{CD}e_C^{\rho}e_D^{\mu}\right) \tag{96}$$

whith

$$e_C^{\rho} = \frac{X_{,C}^{\rho}}{\sqrt{F}}. \tag{97}$$

For a weak deformation of the cylindrical symmetry ($l \gg 1$), a rough approximation gives:

$$e_C^x \sim e_C^y \sim 0 \quad , \quad e_0^t \sim e_3^z \sim 1 \tag{98}$$

So, from (96) the leading terms are tranverse ($i = x, y$):

$$K_{AB}^i \approx \partial_A \partial_B X^i, \tag{99}$$

and the equations of motion of the core are approximated by

$$S^{AB} \partial_A \partial_B X^i = 0. \tag{100}$$

These equations are finally transformed into:

$$A\partial_t^2 X + 2B\partial_t \partial_z X + C\partial_z^2 X = 0 \tag{101}$$

where the constant coefficients are

$$A = \left(\frac{1}{c_s^2} - 1 \right) E^2 - \mu_0^2 \tag{102}$$

$$B = \left(\frac{1}{c_s^2} - 1 \right) EL \tag{103}$$

$$C = \left(\frac{1}{c_s^2} - 1 \right) L^2 + \mu_0^2 \tag{104}$$

with

$$\mu_0^2 = E^2 - L^2 - \frac{M^2}{r_0^2}$$

This equation is of hyperbolic type if $B^2 - AC > 0$.A necessary but not sufficient condition for that is

$$c_s > \frac{1}{\sqrt{2}} \tag{105}$$

This is a necessary condition to have propagating long waves $l \gg 1$ along the core. By a Lorentz tranformation along z we can transform the equation (101) into a classical waves equation. But this can be easily done by choosing the longitudinal momentum $L = 0$. Therefore we can calculate the velocity of propagating long waves which turn out to be above the velocity of light:

$$v_{pr} = \left(\frac{E^2 - \frac{M^2}{r_0^2}}{(2 - \frac{1}{c_s^2})E^2 - \frac{M^2}{r_0^2}} \right)^{\frac{1}{2}} \geq 1. \tag{106}$$

It is not possible to have propagating long waves along the core in this model except in stiff material where the velocity of sound is equal to the velocity of light. In this case the core behaves like a cosmic string.

If $c_s < \frac{1}{\sqrt{2}}$ the equation becomes elliptic and we can have a motion of large amplitude, but our choice of weak pertubation of the cylindrical symmetry does not allow to describe such a situation.

6 Conclusion

These last years the relativistic vortices in superfluid have been analysed in the context of a model based on non-linear wave equations of a complex scalar field. These studies enlight the theoretical similarity between superfluid vortices and global cosmic strings [16, 17, 18, 19, 20].

In this work, we started from a relativistic two-component fluid model and by specifying a stationary cylindrical solution [11] in the form of a relativistic Rankine model, we give the dynamics of the core in the small deviation of the straight configuration. In this way we establish that in stiff matter the equations of motion of the core are the Nambu-Goto equations which are also the equations of the cosmic strings.

Let us note that the purposed model seems to have some agreement with the physical reality where if the core structure of HeII is unknown, there is some experimental evidence that the excitations of the normal fluid tend to cluster near the vortex core [15].

However it is difficult to understand why the long waves cannot propagate along the core of the vortex when the velocity of sound is not equal to c. Since in general the equations are elliptic the amplitude of motion can become large, the method does not allow to study a motion without some severe boundary conditions.

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